GROUP THEORY

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Theorem 1. *for any distinct prime p, q, r every group G of size rpq is solvable*

for the theorem we need the following lemma:

Lemma 2. $\forall$ *prime p, q, r such that $pq = 1 \mod r$ and $r > q > p$. we have $r \neq 1 \mod q$.*

Proof. Assume the contrary, we have $r = 1 \mod q$.

$\exists n, k \in \mathbb{N}$ such that $nq + 1 = r$, and $kr + 1 = pq$. We have

$$k(nq + 1) + 1 = pq$$
$$\Downarrow$$
$$knq + k + 1 = pq \Rightarrow q|k + 1 \Rightarrow k + 1 \geq q \Rightarrow k \geq p$$
$$\Downarrow$$
$$npq + k + 1 \leq pq$$
$$\Downarrow$$
$$k + 1 \leq pq - npq$$
$$\Downarrow$$
$$0 < k + 1 \leq pq(1 - n)$$
$$\Downarrow$$
$$0 < pq(1 - n)$$
$$\Downarrow$$
$$0 < 1 - n$$
$$\Downarrow$$
$$n < 1$$

contradiction □

Proof of the theorem 1. Assume without loss of generality that $r > q > p$.

Assume the contrary, that G is not solvable.

Given a prime L such that $L||G|$ if G has a unic Sylow subgroup of a size L, then it is a normal subgroup and thus G is solvable, so there is no number such that there is 1 Sylow subgroup of size of that number.

We have pq Sylow subgroups of size r, and rp or r Sylow subgroups of size q.

by Sylow third theorem $pq = 1 \mod r$, and if there is rp Sylow subgroups of size q.

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then $rp = 1 \mod q$.

Else $r = 1 \mod q$

But by lemma 2 $r \neq 1 \mod q$ so there rp Sylow subgroups of size q.

every element in a r Sylow subgroup is of order 1 or r every element in a q Sylow subgroup is of order 1 or q

$$n = \text{ the number of elements of order r } = rpq - pq$$
$$m = \text{ the number of elements of order q } = rpq - rp$$

$$|G| \geq n + m$$
$$\Updownarrow$$
$$rpq \geq rpq - pq + rpq - rp$$
$$\Downarrow$$
$$0 \geq rpq - rp - pq > rpq - rp - rp = rp(q-2) > rp(p-2) \geq 0$$

contradiction □